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Particle propagation under an external field and inelastic collisions: existence of stationary states

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Abstract

We provide a proof of existence of stationary states in the three-dimensional Lorentz model for Maxwell particles accelerated by an external field and suffering inelastic collisions. Any non-zero inelasticity is sufficient to stabilize the system, irrespective of the field strength, and an explicit form of the solution is given in the limit of weak inelasticity. The proof extends to the case of particles interacting with a thermostat.

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1. Introduction

The object of this study is the stationary velocity distribution of a particle immersed in a scattering medium and subject to the action of an external field. Owing to the coupling to the field, the particle moves with a constant acceleration **a**. The accelerated motion is interrupted by binary collisions with the scatterers distributed in space with a constant number density ρ.

The analysis performed here is based on the Boltzmann equation. Consequently, the moving particle is supposed to encounter always new scatterers whose steady state has not yet been perturbed by its motion. This leads to a linear kinetic equation for the velocity distribution of the particle. Rather then considering the hard sphere interaction we choose here the so-called Maxwell model: we assume an isotropic scattering, but in contrast to the case of hard spheres, the cross-section is supposed to decrease as the inverse power of the relative velocity of the colliding pair. With this choice, the collision frequency is independent of the energy of the relative motion which greatly simplifies the mathematical problem allowing an effective use of the Fourier transformation. This simplification has already been noticed a long time ago and exploited in a number of publications (see e.g. [1, 2]). An interesting review of the Maxwell model has been given in [3]. We benefit here from the structure of the Maxwell model to prove the existence of nonequilibrium stationary states. In this paper, we consider both the case of immobile scatterers uniformly distributed in space (the Lorentz model) and the case of mobile scatterers in thermal equilibrium. The relevant equations for the stationary states are recalled and presented in various forms in section 2.

The existence of a stationary state requires the possibility for the particle to dissipate the kinetic energy imparted by the external field. In the Lorentz model, the dissipation mechanism is provided by the energy transfer to the internal degrees of freedom phenomenologically modelled by a restitution parameter less than 1. When the particle is in contact with a thermalized fluid, there is an additional relaxation mechanism through collisions with the particles of the thermostat. The main issue is therefore, for the Lorentz model, to prove that the loss of energy due to inelasticity balances the energy absorbed from the external field, thus assuring the stabilization of the system.

For the one-dimensional Lorentz model in an external field, the stationary problem has been studied earlier in relation with runaway processes resulting of the energy imbalance due to the lack of dissipation [4] (and see the references quoted therein). The authors formulate general conditions on the kernel of the collision operator that rule out runaway phenomena and guarantee the existence of a stationary state. Later, the case of inelastic collisions (still in one dimension) was considered specifically and an explicit solution was constructed in terms of the restitution parameter and field strength [5]. In the present paper, we extend the analysis to the three-dimensional Lorentz gas when the cross-section has the Maxwell form (section 3.1). This extension is not straightforward because of the intricate angular dependence brought in by three-dimensional collisions. We prove the existence of the stationary state for any non-zero inelasticity and any field strength, and give the explicit form of the solution in the limit of weak inelasticity (section 3.2). We emphasize the crucial role played by the inelasticity. It is shown for instance in [6] that in one dimension the kinetic energy of the particle grows indefinitely in time. This fact is also illustrated in our paper for the three-dimensional Maxwell gas in equation (27): one sees that the mean kinetic energy diverges in the elastic limit, reflecting the unbounded absorption of energy from the accelerating field ruling out any stationary state.

In section 4 we show that the stationary state of the accelerated particle in a thermalized fluid can be written as a convolution of a Maxwellian distribution with the previously found solution for the Lorentz gas, a remarkably simple structure. Here the stationary state exists even when collisions are elastic since the particle can now dissipate energy in the fluid. When the inelasticity is different from zero, the Maxwellian distribution carries an effective temperature lower than that of the thermostat. This effective temperature is the same as that obtained by the particle undergoing dissipative collisions in absence of an external driving field [7].

2. Kinetic equations

We denote by m and M the mass of the propagating particle and the mass of the scatterer, respectively. Both particles are assumed to be spherically symmetric. At binary collisions, the momentum remains conserved:

$$m\mathbf{v} + M\mathbf{V} = m\mathbf{v}' + M\mathbf{V}'. \tag{1}$$

Here $\mathbf{v}$ and $\mathbf{V}$ denote the precollisional velocities of the particle and of the scatterer, respectively. They are instantaneously transformed into the primed velocities $\mathbf{v}'$ and $\mathbf{V}'$ after collision.

The change in the relative velocity $(\mathbf{v} - \mathbf{V})$ can be conveniently described by considering the unit vector $\hat{\boldsymbol{\sigma}}$ oriented along the line passing through the centres of the colliding pair at the moment of impact. The normal component (velocity of approach) then suffers the reversal and reduction (dissipation)

$$\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v}' - \mathbf{V}') = -\alpha \hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V}), \qquad 0 \leqslant \alpha < 1, \tag{2}$$

whereas the tangential component remains unchanged; α is the restitution parameter.

Combining equations (1) and (2), we obtain the transformation law induced by inelastic collisions

$$\begin{aligned} \mathbf{v}' &= \mathbf{v} - \mu(1+\alpha)[\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})]\hat{\boldsymbol{\sigma}} \\ \mathbf{V}' &= \mathbf{V} + (1-\mu)(1+\alpha)[\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})]\hat{\boldsymbol{\sigma}}, \end{aligned} \tag{3}$$

where

$$\mu = \frac{M}{m+M}.$$

Equation (2) implies that the inverse transformation is simply obtained by replacing α by $1/\alpha$. Hence, the precollisional velocities

$$\begin{aligned} \mathbf{v}'' &= \mathbf{v} - \mu\left(1 + \frac{1}{\alpha}\right)[\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})]\hat{\boldsymbol{\sigma}} \\ \mathbf{V}'' &= \mathbf{V} + (1-\mu)\left(1 + \frac{1}{\alpha}\right)[\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})]\hat{\boldsymbol{\sigma}} \end{aligned} \tag{4}$$

yield after collision the velocities $\mathbf{v}$ and $\mathbf{V}$. Let us also note the relations

$$\alpha \hat{\boldsymbol{\sigma}} \cdot (\mathbf{v}'' - \mathbf{V}'') = -\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V}), \qquad \alpha\, \mathrm{d}\mathbf{v}''\, \mathrm{d}\mathbf{V}'' = \mathrm{d}\mathbf{v}\, \mathrm{d}\mathbf{V}. \tag{5}$$

We consider the linear Boltzmann equation for the stationary velocity distribution $f(\mathbf{v})$ of the propagating particle interacting with a thermal reservoir at temperature T. Assuming the differential cross-section proportional to $|\mathbf{v} - \mathbf{V}|^{-1}$ (the Maxwell model) and denoting the proportionality constant by κ, the stationary state must obey the equation

$$\begin{aligned} \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) = \frac{\kappa}{\lambda} \int \mathrm{d}\mathbf{V} \int \frac{\mathrm{d}\hat{\boldsymbol{\sigma}}}{\pi} [\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})]\theta[\hat{\boldsymbol{\sigma}} \cdot (\mathbf{v} - \mathbf{V})] \\ \times \left\{ \frac{1}{|\mathbf{v}'' - \mathbf{V}''|} \frac{1}{\alpha^2} f(\mathbf{v}'')\phi_T(\mathbf{V}'') - \frac{1}{|\mathbf{v} - \mathbf{V}|} f(\mathbf{v})\phi_T(V) \right\}, \end{aligned} \tag{6}$$

where λ is the mean free path:

$$\lambda^{-1} = \pi\rho(d+D)^2/4, \tag{7}$$

d and D denote the particle and the scatterer diameters, respectively, the Maxwell velocity distribution

$$\phi_T(V) = \left(\frac{M}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{MV^2}{2k_B T}\right) \tag{8}$$

represents the thermal bath, $\theta(x)$ is the unit Heaviside function and k_B is the Boltzmann constant. The left-hand side of (6) is the drift term due to the accelerating field. The collision term (the right-hand side of (6)) is essentially the same as in equation (6) of [7], after modification of the hard sphere cross-section to the Maxwell one, i.e. after dividing by the relative speed (the factors $|\mathbf{v}'' - \mathbf{V}''|^{-1}$ and $|\mathbf{v} - \mathbf{V})|^{-1}$ in the curly brackets)[3]. With the use of relations (5), it can be checked by a straightforward calculation that the collision term in (6) vanishes when integrated over the velocity space in accordance with the conservation of the number of particles.

[3] In contrast to the present notation, in [7] the mass of the test particle (respectively of the fluid particle) is noted by M (respectively m). Also the parameter η (13) is $\mu(1+\alpha)$ in [7].

A particular case of equation (6) is obtained by taking the limit $M \to \infty$ (or $\mu \to 1$) together with $\lim_{T\to 0} \phi_T(V) = \delta(V)$. We then get the kinetic equation for the Lorentz model in which the accelerated gas propagates through the medium filled with infinitely massive immobile scatterers ($T = 0$) (see equation (3) of [5] adapted to the Maxwell cross-section)

$$\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) = \frac{\kappa}{\lambda}\int \frac{\mathrm{d}\hat{\boldsymbol{\sigma}}}{\pi}[\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}]\theta[\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}] \times\left[\frac{1}{|\mathbf{v}-(1+\alpha^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{v})\hat{\boldsymbol{\sigma}}|}\frac{1}{\alpha^2} f(\mathbf{v}-(1+\alpha^{-1})[\hat{\boldsymbol{\sigma}}\cdot\mathbf{v}]\hat{\boldsymbol{\sigma}}) - \frac{1}{|\mathbf{v}|} f(\mathbf{v})\right]. \tag{9}$$

In the general case of $T > 0$, it is convenient to use the relative velocity $\mathbf{w} = \mathbf{v} - \mathbf{V}$ as the integration variable. Rescaling by the factor of α the normal component of the relative velocity (at fixed $\hat{\boldsymbol{\sigma}}$)

$$\hat{\boldsymbol{\sigma}}\cdot\mathbf{w} \to \alpha(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})$$

and using the equality

$$\int \mathrm{d}\hat{\boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}})\theta(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}}) = \pi, \qquad \hat{\mathbf{w}} = \mathbf{w}/|\mathbf{w}|,$$

we rewrite the Boltzmann equation (6) as

$$\mathbf{e}\cdot\frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) = \int \mathrm{d}\mathbf{w}\int \frac{\mathrm{d}\hat{\boldsymbol{\sigma}}}{\pi}(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}})\theta(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}}) f[\mathbf{v}-\mu(1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\hat{\boldsymbol{\sigma}}] \times \phi_T\{\mathbf{v}-\mathbf{w}+[1-\alpha+(1-\mu)(1+\alpha)](\hat{\boldsymbol{\sigma}}\cdot\mathbf{w})\hat{\boldsymbol{\sigma}}\} - f(\mathbf{v}), \tag{10}$$

where

$$\mathbf{e} = \lambda\mathbf{a}/\kappa.$$

An even more convenient form of the kinetic equation (10) is obtained when the collision term is expressed in terms of the unit vector $\hat{\mathbf{n}}$ oriented along the post-collisional relative velocity for elastic collisions. Putting $\alpha = 1$, we infer from (3)

$$\hat{\mathbf{n}} = \hat{\mathbf{w}} - 2(\hat{\mathbf{w}}\cdot\hat{\boldsymbol{\sigma}})\hat{\boldsymbol{\sigma}} \tag{11}$$

with the Jacobian of the transformation $(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}})\theta(\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{w}})\,\mathrm{d}\hat{\boldsymbol{\sigma}}/\pi = \mathrm{d}\hat{\mathbf{n}}/4\pi$.

Equation (10) finally takes the form

$$\mathbf{e}\cdot\frac{\partial}{\partial \mathbf{v}} f(\mathbf{v}) + f(\mathbf{v}) = \int \mathrm{d}\mathbf{w}\int \frac{\mathrm{d}\hat{\mathbf{n}}}{4\pi} f[\mathbf{v}-(1-\eta)(\mathbf{w}-|\mathbf{w}|\hat{\mathbf{n}})]\phi_T[\mathbf{v}-\mathbf{w}+\eta(\mathbf{w}-|\mathbf{w}|\hat{\mathbf{n}})], \tag{12}$$

where we have defined the parameter

$$\eta = 1 - \mu\frac{1+\alpha}{2}. \tag{13}$$

The choice of the Maxwell cross-section allows for simpler forms of the kinetic equations in the Fourier representation. Fourier transforming equation (12) with

$$\tilde{f}(\mathbf{k}) = \int \mathrm{d}\mathbf{v}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{v}} f(\mathbf{v}),$$
$$\tilde{\phi}_T(\mathbf{p}) = \int \mathrm{d}\mathbf{v}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{v}}\phi_T(\mathbf{v}) = \exp\left(-\frac{p^2}{2\beta M}\right) \qquad p = |\mathbf{p}| \tag{14}$$

leads to the convolution

$$(1 - \mathrm{i}\mathbf{e}\cdot\mathbf{k})\tilde{f}(\mathbf{k}) = \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\int \mathrm{d}\mathbf{w}\,\mathrm{e}^{\mathrm{i}(1-\eta)\mathbf{w}\cdot\mathbf{k}}\int \frac{\mathrm{d}\hat{\mathbf{n}}}{4\pi}\,\mathrm{e}^{\mathrm{i}w\hat{\mathbf{n}}\cdot(\eta\mathbf{k}-\mathbf{p})}\tilde{f}(\mathbf{p})\tilde{\phi}_T(\mathbf{k}-\mathbf{p})$$
$$= \int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\left[\int \mathrm{d}\mathbf{w}\,\mathrm{e}^{\mathrm{i}(1-\eta)\mathbf{w}\cdot\mathbf{k}}\int \frac{\mathrm{d}\hat{\mathbf{n}}}{4\pi}\,\mathrm{e}^{\mathrm{i}w\hat{\mathbf{n}}\cdot\mathbf{p}}\right]\tilde{f}(\eta\mathbf{k}+\mathbf{p})\tilde{\phi}_T((1-\eta)\mathbf{k}-\mathbf{p}). \tag{15}$$

Using the formula

$$\int \mathrm{d}\mathbf{w}\, \mathrm{e}^{\mathrm{i}\mathbf{w}\cdot\mathbf{a}} \int \frac{\mathrm{d}\hat{\mathbf{n}}}{4\pi}\, \mathrm{e}^{\mathrm{i}w\hat{\mathbf{n}}\cdot\mathbf{b}} = \frac{2\pi^2}{ab}\delta(b-a), \qquad a = |\mathbf{a}|, \quad b = |\mathbf{b}|, \tag{16}$$

the bracket in (15) becomes $2\pi^2\delta(p-(1-\eta)k)/p^2$, $k = |\mathbf{k}|$, so that performing the p integral gives

$$(1-\mathrm{i}\mathbf{e}\cdot\mathbf{k})\tilde{f}(\mathbf{k}) = \int \frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}\tilde{f}(k(1-\eta)\hat{\mathbf{p}}+\eta\mathbf{k})\tilde{\phi}_T((1-\eta)(\mathbf{k}-k\hat{\mathbf{p}})), \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{p}, \tag{17}$$

which is the final form of the equation.

The case of the Lorentz gas is obtained by letting $T \to 0$ and $M \to \infty$. In this limit, one has $\tilde{\phi}_T \to 1$, $\mu \to 1$ and

$$\eta \to \frac{1-\alpha}{2} \equiv \epsilon$$

(see (13) and (14)) so that

$$(1-\mathrm{i}\mathbf{e}\cdot\mathbf{k})\tilde{f}(\mathbf{k}) = \int \frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}\tilde{f}(k(1-\epsilon)\hat{\mathbf{p}}+\epsilon\mathbf{k}). \tag{18}$$

It could as well be obtained by a direct Fourier transform of (9). In section 3 we prove the existence of the solution to equation (18). The extension to temperatures $T > 0$ together with concluding comments is presented in section 4.

3. The stationary state for the Lorentz model

3.1. Existence of the stationary state

Upon introducing the application $C_{\hat{\mathbf{p}}} : R^3 \to R^3$

$$C_{\hat{\mathbf{p}}}(\mathbf{k}) = k(1-\epsilon)\hat{\mathbf{p}} + \epsilon\mathbf{k}, \tag{19}$$

equation (18) becomes

$$\tilde{f}(\mathbf{k}) = \frac{1}{1-\mathrm{i}\mathbf{e}\cdot\mathbf{k}}\int \frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}\tilde{f}(C_{\hat{\mathbf{p}}}(\mathbf{k})) \tag{20}$$

and can be solved by iterations. Starting with $\tilde{f}_0(\mathbf{k}) = 1/(1-\mathrm{i}\mathbf{e}\cdot\mathbf{k})$, the Nth iteration reads

$$\tilde{f}_N(\mathbf{k}) = \frac{1}{1-\mathrm{i}\mathbf{e}\cdot\mathbf{k}}\int\frac{\mathrm{d}\hat{\mathbf{p}}_1}{4\pi}\cdots\int\frac{\mathrm{d}\hat{\mathbf{p}}_N}{4\pi}\prod_{n=1}^{N}\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e}} \tag{21}$$

and we show below that for fixed $\mathbf{k}$, $\tilde{f}_N(\mathbf{k})$ form a Cauchy sequence. Indeed, for $M > N$, one has

$$\tilde{f}_N(\mathbf{k}) - \tilde{f}_M(\mathbf{k}) = \frac{1}{1-\mathrm{i}\mathbf{e}\cdot\mathbf{k}}\int\frac{\mathrm{d}\hat{\mathbf{p}}_1}{4\pi}\cdots\int\frac{\mathrm{d}\hat{\mathbf{p}}_M}{4\pi}\left[\prod_{n=1}^{N}\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e}}\right.$$
$$\left.\times\left(1-\prod_{n=N+1}^{M}\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e}}\right)\right]. \tag{22}$$

Since $C_{\hat{\mathbf{p}}}(\mathbf{k})$ (19) is real, $|1/(1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e})| \leqslant 1$. Using this inequality repeatedly together with

$$\left|1-\prod_{i=1}^{j}a_i\right| \leqslant \sum_{i=1}^{j}|1-a_i|, \qquad |a_i| \leqslant 1,$$

one finds that the absolute value of the square bracket in (22) is less than

$$\sum_{n=N+1}^{M}\left|1-\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e}}\right| \leqslant \sum_{n=N+1}^{M}\left|C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\cdot\mathbf{e}\right| \leqslant e\sum_{n=N+1}^{M}\left|C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\right|. \tag{23}$$

Thus,

$$|\tilde{f}_N(\mathbf{k})-\tilde{f}_M(\mathbf{k})| \leqslant e\sum_{n=N+1}^{M}\int\frac{\mathrm{d}\hat{\mathbf{p}}_1}{4\pi}\cdots\int\frac{\mathrm{d}\hat{\mathbf{p}}_n}{4\pi}\left|C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\right| \leqslant e\sum_{n=N+1}^{M}\left(\int\frac{\mathrm{d}\hat{\mathbf{p}}_1}{4\pi}\cdots\int\frac{\mathrm{d}\hat{\mathbf{p}}_n}{4\pi}\left|C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\mathbf{k})\right|^2\right)^{1/2}, \tag{24}$$

where the last inequality follows from the Schwarz inequality and the fact that the angular measures $\mathrm{d}\hat{\mathbf{p}}_j/4\pi$ are normalized to 1. One calculates from (19) that $C_{\hat{\mathbf{p}}}(\mathbf{k})$ contracts the norm of the vector $\mathbf{k}$

$$\int\frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}|C_{\hat{\mathbf{p}}}(\mathbf{k})|^2 = k^2((1-\epsilon)^2+\epsilon^2) < k^2, \tag{25}$$

implying

$$|\tilde{f}_N(\mathbf{k})-\tilde{f}_M(\mathbf{k})| \leqslant ek\sum_{n=N+1}^{M}\left(\sqrt{(1-\epsilon)^2+\epsilon^2}\right)^n. \tag{26}$$

This shows that $\tilde{f}_N(\mathbf{k})$ is a Cauchy sequence for all $0<\epsilon\leqslant\frac{1}{2}\,(0\leqslant\alpha<1)$; thus $\lim_{N\to\infty}\tilde{f}_N(\mathbf{k}) = \tilde{f}(\mathbf{k})$ exists and represents the solution of the stationary equation (9) in Fourier form, up to a multiplicative constant (since the equation is linear). This constant is fixed by the normalization condition $\tilde{f}(0) = \int\mathrm{d}\mathbf{v}\, f(\mathbf{v}) = 1$, and under this condition the solution is unique. Note that the convergence is uniform with respect to $\mathbf{k}$ in compact sets; hence $\tilde{f}(\mathbf{k})$ is a continuous function of $\mathbf{k}$. It is thus not vanishing identically for $\mathbf{k}\neq 0$ since $\tilde{f}(0)\neq 0$. One can remark that a stationary state exists for all values of the field strength and of the inelasticity, in particular for an arbitrarily strong field and arbitrarily weak inelasticity.

3.2. The elastic limit

An analytic form of the distribution can be found in the limit of weak dissipation $\alpha\to 1, \epsilon\to 0$. For this we could use the (singular) perturbative scheme developed in section 4 of [5]. Here, we rather extract directly the weak dissipation behaviour from the iterative solution presented in the preceding subsection. The first moments of the velocity can easily be computed from equation (9) with the results

$$\begin{aligned}\langle\hat{\mathbf{e}}\cdot\mathbf{v}\rangle &= \frac{2e}{1+\alpha}, \qquad \hat{\mathbf{e}} = \frac{\mathbf{e}}{e}\\ \langle|\mathbf{v}|^2\rangle &= \langle\hat{\mathbf{e}}\cdot\mathbf{v}\rangle\frac{4e}{1-\alpha^2}\sim\frac{e^2}{\epsilon}, \qquad \epsilon\to 0.\end{aligned} \tag{27}$$

One sees that the average kinetic energy diverges in the elastic limit as $1/\epsilon$, suggesting that the distribution of the scaled velocity $\mathbf{v}/\sqrt{\epsilon}$ might have a limit as $\epsilon\to 0$. Equivalently, one is led to consider the distribution of the scaled Fourier variable $\sqrt{\epsilon}\mathbf{k}$:

$$\tilde{f}(\sqrt{\epsilon}\mathbf{k}) = \frac{1}{1-\mathrm{i}\sqrt{\epsilon}\mathbf{e}\cdot\mathbf{k}}\lim_{N\to\infty}\int\frac{\mathrm{d}\hat{\mathbf{p}}_1}{4\pi}\cdots\int\frac{\mathrm{d}\hat{\mathbf{p}}_N}{4\pi}\prod_{n=1}^{N}\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\sqrt{\epsilon}\mathbf{k})\cdot\mathbf{e}}. \tag{28}$$

Note that $C_{\hat{\mathbf{p}}}(\sqrt{\epsilon}\mathbf{k}) = \sqrt{\epsilon}(1-\epsilon)k\hat{\mathbf{p}} + \epsilon^{3/2}\mathbf{k}$, implying

$$C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\sqrt{\epsilon}\mathbf{k})\cdot\mathbf{e} = \sqrt{\epsilon}(1-\epsilon)^n k\hat{\mathbf{p}}_n\cdot\mathbf{e} + \mathcal{O}(\epsilon^{3/2})$$
$$\frac{1}{1-\mathrm{i}C_{\hat{\mathbf{p}}_n}\cdots C_{\hat{\mathbf{p}}_1}(\sqrt{\epsilon}\mathbf{k})\cdot\mathbf{e}} = 1 + \mathrm{i}\sqrt{\epsilon}(1-\epsilon)^n k\hat{\mathbf{p}}_n\cdot\mathbf{e} - \epsilon(1-\epsilon)^{2n}k^2(\hat{\mathbf{p}}_n\cdot\mathbf{e})^2 + \mathcal{O}(\epsilon^{3/2}), \tag{29}$$

where we have expanded the fraction up to order ϵ. The infinite product in (28), after taking (29) into account (omitting the $\mathcal{O}(\epsilon^{3/2})$ corrections) and performing the $\mathrm{d}\hat{\mathbf{p}}_j$ integrals, takes the form

$$\prod_{n=1}^{\infty}\left(1-\epsilon(1-\epsilon)^{2n}\frac{k^2e^2}{3}\right) \sim \exp\left(-\frac{k^2e^2}{3}\epsilon\sum_{n=1}^{\infty}(1-\epsilon)^{2n}\right)$$
$$= \exp\left(-\frac{k^2e^2}{3}\frac{\epsilon(1-\epsilon)^2}{1-(1-\epsilon)^2}\right) \sim \exp\left(-\frac{k^2e^2}{6}\right)(1+\mathcal{O}(\epsilon)). \tag{30}$$

Finally, keeping the non-spherically symmetric effect of the external field at lowest order $\sqrt{\epsilon}$ in (28), one obtains

$$\tilde{f}(\sqrt{\epsilon}\mathbf{k}) = (1+\mathrm{i}\sqrt{\epsilon}\mathbf{e}\cdot\mathbf{k})\exp\left(-\frac{k^2e^2}{6}\right)(1+\mathcal{O}(\epsilon)). \tag{31}$$

By inverse Fourier transform, the asymptotic (non-scaled) velocity distribution as $\epsilon \to 0$ is therefore found to be

$$f(\mathbf{v}) \sim \left(1-\mathbf{e}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(\frac{3\epsilon}{2\pi e^2}\right)^{3/2}\exp\left(-\frac{3\epsilon v^2}{2e^2}\right). \tag{32}$$

One sees that the spherically symmetric part of the distribution becomes Gaussian as $\epsilon \to 0$. This is a consequence of the Maxwell form of the cross-section, in contrast with the hard sphere case where the asymptotic distribution was found to behave as $\exp(-\mathrm{const}\,\epsilon v^4)$ [5].

4. Accelerated particle in a thermalized fluid

We come back to the stationary equation (17) for the velocity distribution of the accelerated Maxwell particle undergoing collisions in a thermal bath. Setting as before

$$C_{\hat{\mathbf{p}}}(\mathbf{k}) = k(1-\eta)\hat{\mathbf{p}} + \eta\mathbf{k}, \tag{33}$$

we write equation (17) in the form

$$\tilde{F}(\mathbf{k}) = \frac{1}{1-\mathrm{i}\mathbf{e}\cdot\mathbf{k}}\int\frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}\tilde{F}(C_{\hat{\mathbf{p}}}(\mathbf{k})\exp\left[-\frac{|\mathbf{k}-C_{\hat{\mathbf{p}}}(\mathbf{k})|^2}{2\beta M}\right]. \tag{34}$$

The notation $\tilde{F}(\mathbf{k})$ is used to distinguish the $T > 0$ distribution from the distribution $\tilde{f}(\mathbf{k})$ constructed in the preceding section for the Lorentz gas. In view of the identity

$$|\mathbf{k}-C_{\hat{\mathbf{p}}}(\mathbf{k})|^2 = \frac{1-\eta}{\eta}(k^2-|C_{\hat{\mathbf{p}}}(\mathbf{k})|^2), \tag{35}$$

equation (34) can be cast in the form

$$\tilde{F}(\mathbf{k}) = \exp\left(-\frac{k^2}{2\beta_{\mathrm{eff}}m}\right)\frac{1}{1-\mathrm{i}\mathbf{e}\cdot\mathbf{k}}\int\frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}\tilde{F}(C_{\hat{\mathbf{p}}}(\mathbf{k}))\exp\left(\frac{|C_{\hat{\mathbf{p}}}(\mathbf{k})|^2}{2\beta_{\mathrm{eff}}m}\right), \tag{36}$$

where we have defined an effective inverse temperature by

$$\beta^{\mathrm{eff}} = \frac{M\eta}{m(1-\eta)}\beta = (k_B T_{\mathrm{eff}})^{-1}. \tag{37}$$

It is now clear that the function $\exp(k^2/2\beta_{\mathrm{eff}}m)\tilde{F}(\mathbf{k})$ obeys equation (20) for the Lorentz gas with ϵ replaced by η. Therefore, by the same analysis as in section 3.1, there exists a solution to equation (36) of the form

$$\tilde{F}(\mathbf{k}) = \exp\left(-\frac{k^2}{2\beta_{\mathrm{eff}}m}\right)\tilde{f}(\mathbf{k}), \tag{38}$$

where $\tilde{f}(\mathbf{k}) = \lim_{N\to\infty}\tilde{f}_N(\mathbf{k})$ is the limit of the successive iterations (21) constructed for the Lorentz gas with parameter η. Thus, in velocity space the solution is simply a convolution of the Lorentz gas-type distribution $f(\mathbf{v})$ with a Maxwellian at effective temperature T_{eff}:

$$F(\mathbf{v}) = \int \mathrm{d}\mathbf{v}'\phi_{T_{\mathrm{eff}}}(\mathbf{v}-\mathbf{v}')f(\mathbf{v}'). \tag{39}$$

Equation (39) establishes a remarkably simple relation between the stationary state f in a zero temperature Lorentz model and the corresponding state F in a thermostat. Since η is strictly positive as long as the temperature of the host fluid is different from 0 (irrespective of the value of the inelasticity), see (13), the distribution $F(\mathbf{v})$ is well defined for all $0 \leqslant \alpha \leqslant 1$ and all values of the field strength e.

The effective temperature (37) characterizes the stationary state in the absence of electric field as already shown in [7] for a large class of cross-sections (for the notation, see footnote 3). Indeed, when $e = 0$, $\tilde{f}(\mathbf{k}) = 1$, $f(\mathbf{v}) = \delta(\mathbf{v})$, then $F(\mathbf{v})\phi_{T_{\mathrm{eff}}}(\mathbf{v})$ reduces to a Maxwellian distribution with the effective temperature $T_{\mathrm{eff}} < T$. This fact is a consequence of mathematical equivalence of the dynamics with inelastic collisions with the elastic case with an effective mass at the level of the Boltzmann kinetic theory [8].

For $e \neq 0$, the field dependence is entirely contained in $f(\mathbf{v})$. The linear response part of it is obtained by expanding the fractions in (21) to linear order in e yielding, with $\int \frac{\mathrm{d}\hat{\mathbf{p}}}{4\pi}C_{\hat{\mathbf{p}}}(\mathbf{k}) = \eta k$,

$$\tilde{f}(\mathbf{k}) \sim 1 + \mathrm{i}\mathbf{e}\cdot\mathbf{k} + \mathrm{i}\mathbf{e}\cdot\mathbf{k}\sum_{n=1}^{\infty}\eta^n = 1 + \frac{\mathrm{i}\mathbf{e}\cdot\mathbf{k}}{1-\eta}, \qquad e \to 0. \tag{40}$$

When this is inserted in (38), (39) one finds the modification of the velocity distribution to first order in the field

$$F(\mathbf{v}) \sim \left(1 - \left(\frac{1}{1-\eta}\right)\mathbf{e}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\phi_{T_{\mathrm{eff}}}(\mathbf{v}). \tag{41}$$

Thus, the mean velocity at order e equals

$$\langle\mathbf{v}\rangle \sim \frac{\mathbf{e}}{1-\eta}, \tag{42}$$

which agrees with (27) in the Lorentz model limit $M \to \infty$. Note that the current (42) is temperature independent, a peculiarity of the Maxwell gas.

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